Comment about HW:

1. Arguing the existence of limit by defition is usually tedious and boring. You should know how to do it when you are asked to. In practice there are many other ways to use. Make sure you're familiar with the theorems & propositions in the book. It's safe to know how to prove.

2. $\lim_{x\to 0} x\cdot\cos\frac{1}{x}$ exists and is zero.

The quickiest way:

Note that if $\lim_{x\to c} f(x)=0$, $g(x)$ is bounded near $x=c$.

then $\lim_{x\to c} f(x)g(x) = 0$. (Thm 2.6).

Pf of Thm: $\forall\, \varepsilon>0,\ \overset{?}{\exists}\,\delta>0.\ \forall\, 0<|x-c|<\delta,\ |f(x)g(x)|<\varepsilon$.

$g(x)$ bdd near $x=c \Rightarrow \exists M>0, \exists\,\delta_1>0,\ |x-c|<\delta_1,\ |g(x)|\le M$ ①

$f(x)\to 0$ as $x\to 0 \Rightarrow |f(x)|$ becomes arbitrarily small when x becomes sufficiently close to c.

$\forall\, \varepsilon>0,\ \exists\,\delta_2>0,\ \forall\, 0<|x-c|<\delta_2,\ |f(x)|<\frac{\varepsilon}{M}$ ②

Pick $\delta=\min(\delta_1,\delta_2)$. So $0<|x-c|<\delta$, both ①, ② hold.

This tells me: $0<|x-c|<\delta$

$$\Rightarrow |f(x)g(x)| = |f(x)|\cdot|g(x)| \overset{\text{b/c ①}}{<} \frac{\varepsilon}{M}|g(x)| \overset{\text{b/c ②}}{<} \frac{\varepsilon}{M}\cdot M = \varepsilon.$$

Topic 1: One-sided limits.

Left-hand limit: $\lim_{x\to c^-} f(x) = L \Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0, \forall x \in (c-\delta, c),$
$|f(x) - L| < \varepsilon.$

Right-hand limit: $\lim_{x\to c^+} f(x) = L \Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0, \forall x \in (c, c+\delta)$
$|f(x) - L| < \varepsilon.$

Thm: $\lim_{x\to c} f(x) = L \Leftrightarrow \lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = L.$

Pf: $(\Rightarrow)$.

$\lim_{x\to c} f(x) = L \Rightarrow \forall \varepsilon > 0, \exists \delta > 0, \forall x \in D$
$0 < |x-c| < \delta \Rightarrow |f(x) - L| < \varepsilon$

In particular, $x \in (c-\delta, c) \Rightarrow |f(x) - L| < \varepsilon.$

This tells $\lim_{x\to c^-} f(x) = L$. Similarly (try it yourself). $\lim_{x\to c^+} f(x) = L.$

$(\Leftarrow)$. $\lim_{x\to c^-} f(x) = L \Rightarrow \forall \varepsilon > 0, \exists \delta > 0, \forall x \in (c-\delta, c), |f(x) - L| < \varepsilon$

$\lim_{x\to c^+} f(x) = L \Rightarrow \forall \varepsilon > 0, \exists \delta > 0, \forall x \in (c, c+\delta), |f(x) - L| < \varepsilon.$

Not so fast! ~~So~~ $\forall \varepsilon > 0, \exists \delta > 0, \forall x \in (c-\delta, c) \cup (c, c+\delta), |f(x) - L| < \varepsilon$

Those δ's may differ!

Hence $\lim_{x\to c} f(x) = L.$

Modification: $\lim_{x\to c^-} f(x) = L \Rightarrow \forall \varepsilon > 0, \exists \delta_1 > 0, \forall x \in (c-\delta_1, c), |f(x) - L| < \varepsilon$ ①

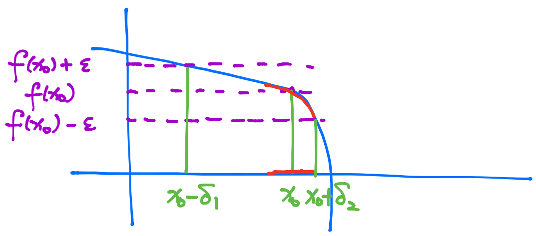
$\lim_{x\to c^+} f(x) = L \Rightarrow \forall \varepsilon > 0, \exists \delta_2 > 0, \forall x \in (c, c+\delta_2), |f(x) - L| < \varepsilon.$ ②

Pick $\delta = \min(\delta_1, \delta_2)$, so $\forall x \in (c-\delta, c) \cup (c+\delta, c)$
both ①, ② hold.

So $\forall \varepsilon > 0, \exists \delta > 0, \forall x \in (c-\delta, c) \cup (c, c+\delta), |f(x) - L| < \varepsilon$

Hence $\lim_{x \to c} f(x) = L$.

Illustration



δ_1 and δ_2 may be very different.

But within $(x_0-\delta, x_0+\delta)$, $\delta = \min(\delta_1, \delta_2)$

we still have $|f(x) - f(x_0)| < \varepsilon$.

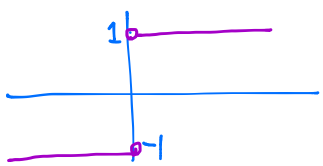
Give examples of functions, s.t.

(1) $\lim_{x \to c^-} f(x)$ exists. $\lim_{x \to c^+} f(x)$ exists. not equal.

(2) $\lim_{x \to c^-} f(x)$ exists. $\lim_{x \to c^+} f(x)$ DNE

(3). $\lim_{x \to c^-} f(x)$ DNE $\lim_{x \to c^+} f(x)$ DNE.

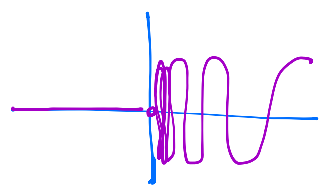
(1)



$f(x) = \frac{x}{|x|}$

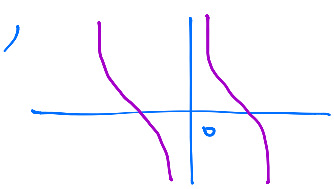
jump discontinuity.

(2)



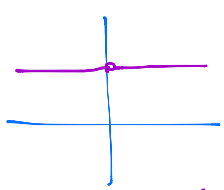
$f(x) = \begin{cases} 0 & x < 0 \\ \sin\frac{1}{x} & x > 0. \end{cases}$

(3)



$f(x) = \cot x$

essential discontinuity.



$f(x) = \frac{x}{x}. = \begin{cases} 1 & x \neq 0. \\ \text{undefined} & x = 0. \end{cases}$

removable discontinuity.

Topic 2: limit at infinity.

$\lim_{x \to +\infty} f(x) = L$ if $\forall \varepsilon > 0, \exists M > 0, \forall x > M, |f(x) - L| < \varepsilon.$

$\lim_{x \to -\infty} f(x) = L$ if $\forall \varepsilon > 0, \exists M > 0, \forall x < -M, |f(x) - L| < \varepsilon.$

Example: $f(x) = \frac{\sqrt{x}}{x+2}$

(1) Guess $\lim_{x \to +\infty} f(x) = 0$

(2) Prove it.

For $\varepsilon > 0$, pick $M = \frac{1}{\varepsilon^2}$. For $x > M$

$$\left|\frac{\sqrt{x}}{x+2}\right| = \frac{\sqrt{x}}{x+2} < \frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}} < \frac{1}{\sqrt{M}} = \varepsilon.$$

by knowledge of ineq.

$\square$.

Example: $f(x) = \frac{\sqrt{x}}{x-2}$

(1) Guess $\lim_{x \to +\infty} f(x) = 0$

(2) Prove it.

Let $M_1 = 2$, so for $x > M_1$, $x - 2 > 0 \Rightarrow \left|\frac{\sqrt{x}}{x-2}\right| = \frac{\sqrt{x}}{x-2}$ ①

(Idea: $x \gg 2$, I can make $-2 > -\frac{1}{2}x \Rightarrow x > 4$)

Let $M_2 = 4$, so for $x > M_2$, $-2 < -\frac{1}{2}x. \Rightarrow x - 2 > x - \frac{1}{2}x = \frac{1}{2}x$

$$\frac{\sqrt{x}}{x-2} < \frac{\sqrt{x}}{\frac{1}{2}x} = \frac{2}{\sqrt{x}}. \quad ②$$

Let $M_3 = \frac{4}{\varepsilon^2}$. so for $x > M_3$, $\frac{2}{\sqrt{x}} < \frac{2}{\sqrt{\frac{4}{\varepsilon^2}}} = \varepsilon.$ ③

So $\forall \varepsilon > 0$, pick $M = \max(M_1, M_2, M_3)$, so $x > M$, all ①, ②, ③ hold

$$\left|\frac{\sqrt{x}}{x-2}\right| \underset{①}{=} \frac{\sqrt{x}}{x-2} \underset{②}{<} \frac{2}{\sqrt{x}} \underset{③}{<} \varepsilon$$

$\square$.

Exercise: (1) $\dfrac{\sqrt{x}}{x - 1000000} \longrightarrow 0$ as $x \to +\infty$.

Workshop: Project 2.4. Problem 3. Submit next week.